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Higgs Field–Fermion Coupling in the Tensor Dirac Theory

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In previous work, the Dirac and Einstein equations were unified in a tetrad formulation of a Kaluza–Klein model which gives precisely the usual Dirac–Einstein Lagrangian. In this model, the self-adjoint modes of the tetrad describe gravity, whereas the isometric modes of the tetrad together with a scalar field describe fermions. The tetrad Kaluza–Klein model is based on a constrained Yang–Mills formulation of the Dirac Lagrangian in which the bispinor field Ψ is mapped to a set of $SL(2, R) \times U(1)$ gauge potentials A^K_α and a complex scalar field ρ. In this paper we generalize the map $\Psi \rightarrow (A^K_\alpha, \rho)$ to multiplets of n bispinor fields representing a fermion multiplet as in standard electroweak theory. We show that the Lagrangian for bispinor multiplets used in the Standard Model becomes a constrained Yang–Mills Lagrangian, for which the Higgs field determines a noninvariant gauge metric, thereby breaking the full gauge symmetry.

1. INTRODUCTION

In previous work, the Dirac and Einstein equations were unified in a tetrad formulation of a Kaluza–Klein model which gives precisely the usual Dirac–Einstein Lagrangian [1, 2]. In this model, the self-adjoint modes of the tetrad describe gravity, whereas the isometric modes of the tetrad together with a scalar field describe fermions. An analogy can be made between the tetrad modes and the elastic and rigid modes of a deformable body [1]. For a deformable body, the elastic modes are self-adjoint and the rigid modes are isometric with respect to the Euclidean metric on R^3. This analogy extends into the quantum realm since rigid modes satisfying Euler's equation can be Fermi quantized [3].

The tetrad Kaluza–Klein model is based on a constrained Yang–Mills formulation of the Dirac theory [1–4]. In this formulation a bispinor field

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Ψ is mapped to a set of $SL(2, R) \times U(1)$ gauge potentials A_α^K and a complex scalar field ρ. The map $\Psi \rightarrow (A_\alpha^K, \rho)$ imposes an orthogonal constraint on the gauge potentials A_α^K which is explicated by the use of tetrads in the Kaluza–Klein model [1, 2]. Via this map, the Dirac bispinor Lagrangian equals the constrained Yang–Mills Lagrangian in the limit of an infinitely large coupling constant [3, 4]. This limit in the Kaluza–Klein model is equivalent to the radius of the higher compact dimensions becoming vanishingly small even when compared to the Planck length [1, 2].

In a recent paper the tensor Dirac theory was generalized to the larger $SL(2, C) \times U(1)$ gauge group acting on bispinors, and it was shown that each $SL(2, R) \times U(1)$ subgroup of $SL(2, C) \times U(1)$ corresponds to a different factorization of the second-order Klein–Gordon equation into a first-order Dirac equation [5]. This symmetric formulation, which includes both the Dirac and Majorana bispinor theories as special cases, was previously studied using Clifford algebra techniques [6].

In this paper we generalize further by defining a map from multiplets of n bispinor fields $\Psi = (\Psi_1, \ldots, \Psi_n)$ to tensor fields (A_α^K, ρ_j) where now the A_α^K are $SL(2n, C) \times U(1)$ gauge potentials, and ρ_j for $1 \leq j \leq n$ are complex scalar fields. Apart from the exceptional set where the $\rho_j = 0$, the map $\Psi \rightarrow (A_\alpha^K, \rho_j)$ is a double covering map onto its image. (Such double covering maps have no observable effects [3, 4, 7].) The image of this map contains only gauge potentials A_α^K which are associated with an $SU(n, n) \times U(1)$ subgroup of $SL(2n, C) \times U(1)$, and which satisfy an orthogonal constraint. [In the case $n = 1$, note that $SU(1, 1) \cong SL(2, R)$.] The restriction to $SU(n, n) \times U(1)$ arises because the Dirac equation for bispinor multiplets has only $SU(n, n) \times U(1)$ gauge symmetry, whereas the larger $SL(2n, C) \times U(1)$ gauge group acts on the bispinor multiplets themselves. Noting from Section 5 that electroweak gauge transformations form an $SU(2) \times U(1)$ subgroup of $SU(n, n)$, we show that the tensor fields (A_α^K, ρ_j) give a faithful representation of the fermion sector of the Standard Model (e.g., the first generation of quarks and leptons consisting of an up quark, down quark, neutrino, and electron).

In previous work, we argued that quantum mechanics need only consist of three parts: the classical field equations, field quantization, and rules for applying the formalism to experiments [5]. Field quantization requires the identification of the physically realizable solutions of the classical field equations, construction of a Hilbert space containing them, and definition of field operators, which act on the Hilbert space and which preserve the coherent subspaces defining the superselection rules. The field operators (including the Hamiltonian operator) are thus restricted to act on physically realizable states. As is the case for gauge bosons (e.g., when Feynman or Landau gauge-fixing terms are added to the photon Lagrangian [8]), two classical

Lagrangians are equivalent if all their observables are equal when restricted to physically realizable solutions. For this reason it suffices to consider classical tensor forms of fermion Lagrangians for which all observables equal their bispinor version when restricted to physically realizable solutions. The tensor form of the Standard Model Lagrangian for the fermion sector presented in Section 5 is shown to satisfy this criterion of equivalence.

In the standard electroweak theory [9], the Higgs field selects, by its orientation in the vacuum (i.e., its nonvanishing vacuum expected value), the gauge transformations associated with the massive Z and W bosons and the massless photon. However, we know that electromagnetic interactions conserve parity, whereas weak interactions do not. Thus the parity map must commute with electromagnetic gauge transformations, but not with the parity-violating, weak gauge transformations. In this paper we will show that a one-to-one correspondence exists between the vacuum orientations of the Standard Model Higgs field and parity maps defined on Ψ which is consistent with the selection of the electroweak bosons. That is, selecting the Higgs field orientation is equivalent to choosing the parity map acting on Ψ, which breaks the electroweak gauge symmetry by the following argument.

Because the Dirac gamma matrices γ^δ do not commute with all electroweak gauge transformations, in order to understand the electroweak gauge symmetry of the Standard Model, it is best to write the free-fermion Lagrangian using the original Dirac matrices $\alpha^\delta = \gamma^0\gamma^\delta$ and $\beta = \gamma^0$ as follows [10]:

$$L = \mathrm{Re}[i\Psi^+\alpha^\delta\, \partial_\delta\Psi - \sum_{j=1}^{n} m_j\Psi^+\beta_j\Psi] \tag{1.1}$$

where m_j for $1 \leq j \leq n$ is the jth fermion mass, ∂_δ for $\delta = 0, 1, 2, 3$ denote the space-time partial derivatives acting on the bispinor multiplet field $\Psi = (\Psi_1, \ldots, \Psi_n)$, and Ψ^+ denotes the transpose conjugate of Ψ in bispinor notation. Also, we define the matrices $\beta_j = \beta\pi_j$, where π_j is the projection onto the jth flavor subspace (see Section 5).

Note that $\beta = \sum_{j=1}^{n} \beta_j$ equals the parity map P acting on the bispinor multiplet Ψ. Whereas the matrices α^δ commute with all electroweak gauge transformations, the matrices β and β_j do not. To make the Lagrangian (1.1) invariant for all electroweak gauge transformations, the Standard Model defines the restricted parity maps β_j to be functions of the Higgs field [9]. In Section 5, using the one-to-one correspondence between the Higgs field orientations and the parity maps, we show that selecting the Higgs field is equivalent to choosing the parity map P and hence to selecting the matrices β_j in the Lagrangian (1.1).

Previously only restrictions on the Higgs field arising from interactions with bosons have been explicitly considered [9]. However, the fermion

Lagrangian (1.1) imposes additional constraints on the Standard Model Higgs field. In Section 4 we prove that parity maps are in one-to-one correspondence with unitary symplectic forms. Thus, linking the Higgs field to the parity map constrains the Higgs field to have a symplectic structure from which we can classify all possible Higgs field–fermion couplings as described in Section 5. This property of the Standard Model Higgs field, arising from interaction with fermions as in the Lagrangian (1.1), was not previously recognized.

Also, in the literature there is little if any mention of the effect of the parity map P on the tensor observables associated with a bispinor multiplet Ψ. In Section 5, it is shown by the map $\Psi \to (A^K_\alpha, \rho_j)$ that the bispinor parity map P transforms the tensor fields (A^K_α, ρ_j) as follows:

$$\begin{aligned} A^K_\alpha &\xrightarrow{P} \overline{A}^\alpha_K = g_{JK}\, g^{\alpha\beta}\, \overline{A}^J_\beta \\ \rho_j &\xrightarrow{P} \overline{\rho}_j \end{aligned} \tag{1.2}$$

where the bars denote complex conjugation. Note that in formula (1.2) the space-time index α is lowered and raised using the space-time metric $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$, whereas the gauge index K is lowered and raised using the gauge metric g_{JK} and its inverse g^{JK}. We see from formula (1.2) that both g_{JK} and $g_{\alpha\beta}$ depend on the choice of parity map P. We further show that while for all choices of parity maps $g_{\alpha\beta}$ can only be the Minkowski space-time metric, the gauge metric g_{JK} depends covariantly on the Higgs field, whose vacuum orientation is in one-to-one correspondence with parity maps acting on Ψ. Thus, while the ρ_j are defined as invariant scalar fields, both A^K_α and g_{JK} transform covariantly under electroweak gauge transformations, making the Yang–Mills Lagrangian for (A^K_α, ρ_j) invariant. Broken symmetry is therefore manifested in the tensor Dirac theory by the dependence of the gauge metric g_{JK} on the Higgs field.

In Section 2 we review the derivation which demonstrates that the Dirac bispinor Lagrangian equals a constrained Yang–Mills Lagrangian in the limit of an infinitely large coupling constant. Starting from trace formulas for the Pauli matrices, we derive Fierz identities, first for spinors in Section 3 then for multiplets of $2n$ spinors in Section 4. In Section 5 the tensor form of the fermion Lagrangian in standard electroweak theory is derived. In Section 5 we also discuss the close tie of the Higgs field with the fermion structure, including mass, as well as flavor superselection in the tensor Dirac theory.

Having discussed the equivalence of the tensor and bispinor forms of the fermion Lagrangian used in the Standard Model, we conclude this introduction with some remarks about the importance of the classical tensor forms of fermion Lagrangians to the foundations of quantum field theory.

As previously stated, the tensor form of Dirac's bispinor Lagrangian describes a tetrad together with a scalar field in a Kaluza–Klein model which unifies fermions and gravity. The use of tetrads to describe gravity has a long history [11], which includes coupling with the Dirac field as a source [12]. However, introducing a tetrad to describe both fermion and gravitational fields solves an important problem posed by current theories of fermion–graviton interaction. To define bispinors, reference tetrad fields or their equivalent must be defined on the space-time manifold [13]. In supersymmetric theories these tetrad fields have been treated as purely boson fields with superfluous degrees of freedom [14]. In the Kaluza–Klein tetrad model the reference tetrads themselves are the fundamental fields which describe both fermions and gravity, without superfluous degrees of freedom [1]. This suggests a simplification of the dynamical variables used in quantum gravity [15].

2. TENSOR FORM OF THE DIRAC LAGRANGIAN

In previous papers we derived the tensor form of Dirac's bispinor Lagrangian and reviewed the history of such derivations by Takahashi and others [1, 5, 16]. To introduce the notation needed for the remainder of this paper, we will briefly review in this section the derivation which demonstrates that the Dirac bispinor Lagrangian (2.4) equals the constrained Yang–Mills Lagrangian (2.14) in the limit of an infinitely large coupling constant. (In Kaluza–Klein geometry this limit is equivalent to the radius of the higher compact dimensions being very small compared to the Planck length [1].) In addition, we will show how all bispinor observables (e.g., the energy-momentum tensor $T^{\alpha\beta}$, spin polarization tensor $S^{\alpha\beta\gamma}$, and the electric current vector J_α for the Dirac bispinor field Ψ) can be derived directly from well-known Yang–Mills formulas.

The derivation proceeds from the $SL(2, R) \times U(1)$ gauge symmetry of Dirac's bispinor Lagrangian. Consider the $SL(2, R) \times U(1)$ gauge transformations, acting on the bispinor field Ψ, with infinitesimal generators τ_K for $K =$ 0, 1, 2, 3 defined by

$$\begin{aligned} \tau_0\Psi &= -i\Psi, \qquad & \tau_1\Psi &= i\Psi^C \\ \tau_2\Psi &= \Psi^C, \qquad & \tau_3\Psi &= i\gamma^5\Psi \end{aligned} \tag{2.1}$$

where (using bispinor notation) Ψ^C denotes the charge conjugate of Ψ and γ^5 is the fifth Dirac matrix [10]. Note that the action of $SL(2, R) \times U(1)$ on Ψ is real linear, whereas usually only complex linear gauge transformations of bispinors are considered. The infinitesimal gauge generators τ_0, τ_1, and τ_2 generate $SL(2, R)$ and τ_3 generates $U(1)$.

The $SL(2, R) \times U(1)$ gauge transformations generated by τ_K commute with Lorentz transformations [10]. From formula (2.1) the commutation relations of the gauge generators τ_K are given by

$$
\begin{aligned}
[\tau_0, \tau_1] &= 2\tau_2 \\
[\tau_0, \tau_2] &= -2\tau_1 \\
[\tau_1, \tau_2] &= -2\tau_0
\end{aligned}
\tag{2.2}
$$

and τ_3 commutes with all the τ_K. Formula (2.2) can be written more compactly as

$$
[\tau_J, \tau_K] = 2f^L_{JK}\tau_L \tag{2.3}
$$

which defines the Lie algebra structure constants f^L_{JK} for the gauge group $SL(2, R) \times U(1)$.

By formula (2.2), the Minkowski metric g_{JK} (with diagonal elements $\{1, -1, -1, -1\}$ and zeros off the diagonal) is an invariant metric [17] for the gauge group $SL(2, R) \times U(1)$. Gauge indices J, K, L will be lowered and raised using the Minkowski metric g_{JK} and its inverse g^{JK}. As in formula (2.3), repeated indices are to be summed from 0 to 3.

Dirac's bispinor Lagrangian L is given by

$$
L = \mathrm{Re}[i\overline{\Psi}\gamma^\alpha\, \partial_\alpha \Psi - ms] \tag{2.4}
$$

where s is the complex scalar field defined by

$$
\begin{aligned}
\mathrm{Re}[s] &= \overline{\Psi}\Psi \\
\mathrm{Im}[s] &= i\overline{\Psi}\gamma^5\Psi
\end{aligned}
\tag{2.5}
$$

where (using bispinor notation) $\overline{\Psi} = \Psi^+\gamma^0$, where Ψ^+ denotes the transpose conjugate of Ψ, and γ^α for $\alpha = 0, 1, 2, 3$ are Dirac matrices [10]. Moreover, in formula (2.4), m denotes the fermion mass and ∂_α denote partial derivatives with respect to space-time coordinates. Tensor indices α, β, γ are lowered and raised using the Minkowski space-time metric, which we denote as $g_{\alpha\beta}$, and its inverse $g^{\alpha\beta}$.

Apart from the mass term, Dirac's bispinor Lagrangian is invariant under the $SL(2, R) \times U(1)$ gauge transformations (2.1). From formula (2.5) the scalar s is invariant under $SL(2, R)$ gauge transformations, and transforms as a complex scalar under the $U(1)$ gauge transformations generated by τ_3. To make the Lagrangian (2.4) invariant for all $SL(2, R) \times U(1)$ gauge transformations, it suffices that m transform like $\bar{s}$ (the complex conjugate of s). Since m appears in the Lagrangian (2.4) without derivatives, the assumption that m transform like $\bar{s}$ under $U(1)$ gauge transformations has no effect on the Dirac equation.

From the Dirac Lagrangian (2.4) we can derive the following $SL(2, R) \times U(1)$ Noether currents:

$$j^K_\alpha = \mathrm{Re}[i\overline{\Psi}\gamma_\alpha\tau^K\Psi] \tag{2.6}$$

The Noether currents j^K_α and scalar s satisfy an orthogonal constraint, a Fierz identity [5, 16] (this will be generalized in Section 4 for bispinor multiplets):

$$j^K_\alpha j_{K\beta} = |s|^2 g_{\alpha\beta} \tag{2.7}$$

Takahashi [16] derived the following formula for the kinetic part of the Dirac Lagrangian (2.4):

$$\mathrm{Re}[i\overline{\Psi}\gamma^\alpha\partial_\alpha\Psi] = -\frac{1}{4|s|^2}\,\mathrm{Re}[(\partial_\alpha\mathbf{J}_\beta)\cdot\mathbf{J}^\alpha\times\mathbf{J}^\beta - 2i\bar{s}J^0_\alpha\,\partial^\alpha s] \tag{2.8}$$

which uses the following notation (which differs from Takahashi's):

$$J^K_\alpha = (J^0_\alpha, \mathbf{J}_\alpha) = (-j^3_\alpha, -ij^2_\alpha, ij^1_\alpha, -j^0_\alpha) \tag{2.9}$$

Thus, from Takahashi's formula (2.8), we see that the Dirac Lagrangian (2.4) can be expressed entirely in terms of the Noether currents j^K_α and the complex scalar field s, satisfying the orthogonal constraint (2.7). Formula (2.8) is derived from first principles in ref. 5. Once the $SL(2, R) \times U(1)$ gauge symmetry of formula (2.8) is recognized, the demonstration that Dirac's bispinor Lagrangian (2.4) equals a constrained Yang–Mills Lagrangian in the limit of an infinitely large coupling constant is fairly obvious.

Indeed, we can map a subset of $SL(2, C) \times U(1)$ gauge potentials A^K_α and a complex scalar field ρ into (J^K_α, s) by setting

$$\begin{aligned} J^K_\alpha &= 4|\rho|^2 A^K_\alpha \\ s &= 4|\rho|^2\bar{\rho} \end{aligned} \tag{2.10}$$

Since we regard the Lie algebra of $SL(2, C)$ as the complexification of the Lie algebra of $SU(2)$, the $SL(2, C)$ gauge potentials $\mathbf{A}_\alpha = (A^1_\alpha, A^2_\alpha, A^3_\alpha)$ are complex, while the $U(1)$ gauge potential A^0_α is real. By formula (2.9) the gauge potentials A^K_α are restricted to the subset for which

$$\mathrm{Re}[A^1_\alpha] = \mathrm{Re}[A^2_\alpha] = \mathrm{Im}[A^3_\alpha] = 0 \tag{2.11}$$

This subset corresponds precisely to an $SL(2, R) \times U(1)$ subgroup of the gauge group $SL(2, C) \times U(1)$. On substituting formula (2.10) into Takahashi's formula (2.8), Dirac's Lagrangian (2.4) becomes

$$L = -\mathrm{Re}[(\partial_\alpha\,\mathbf{A}_\beta)\cdot\mathbf{A}^\alpha\times\mathbf{A}^\beta + 2i\bar{\rho}\,A^0_\alpha\,\partial^\alpha\rho + 4m\,|\rho|^2\bar{\rho}] \tag{2.12}$$

and the orthogonal constraint (2.7) becomes

$$A^K_\alpha A_{K\beta} = -|\rho|^2 g_{\alpha\beta} \tag{2.13}$$

Consider the following Yang–Mills Lagrangian L_g for the gauge potentials A^K_α and the complex scalar field ρ:

$$L_g = -\tfrac{1}{4}\mathrm{Re}[A^K_{\alpha\beta}A^{\alpha\beta}_K] + \overline{D_\alpha(\rho + \mu)}D^\alpha(\rho + \mu) - \tfrac{1}{2}g^2|\rho|^4 \tag{2.14}$$

where $A^K_{\alpha\beta} = (A^0_{\alpha\beta}, \mathbf{A}_{\alpha\beta})$ and

$$\begin{aligned} A^0_{\alpha\beta} &= \partial_\alpha A^0_\beta - \partial_\beta A^0_\alpha \\ \mathbf{A}_{\alpha\beta} &= \partial_\alpha \mathbf{A}_\beta - \partial_\beta \mathbf{A}_\alpha - g\mathbf{A}_\alpha \times \mathbf{A}_\beta \\ D_\alpha(\rho + \mu) &= \partial_\alpha\rho + igA^0_\alpha(\rho + \mu) \end{aligned} \tag{2.15}$$

where g denotes the Yang–Mills coupling constant and $\mu = (2/g)m$, where m is the fermion mass. From formulas (2.12) and (2.13), Dirac's bispinor Lagrangian (2.4) equals

$$L = \lim_{g\to\infty} g^{-1}L_g \tag{2.16}$$

Note that the Euler–Lagrange equation for the Lagrangian (2.14) with the constraint (2.13) expressed using Lagrange multipliers commutes with the restriction (2.11). Hence, the $\mathbf{A}_\alpha$ can be used to denote either $SL(2, C)$ or the subset of $SL(2, R)$ gauge potentials. By regarding $SL(2, R)$ as embedded in the complex analytic group $SL(2, C)$, we are able to use familiar vector operations to express the Lie algebra structure constants in formulas (2.12) and (2.15). The vector operations greatly simplify derivations.

Note also from the Lagrangian (2.16) that we can derive all bispinor observables (e.g., the energy-momentum tensor $T^{\alpha\beta}$, spin polarization tensor $S^{\alpha\beta\gamma}$, and the electric current vector J^α) directly from the Yang–Mills formulas. For example, the Dirac spin polarization tensor $S^{\alpha\beta\gamma}$ is usually expressed in bispinor notation as

$$S^{\alpha\beta\gamma} = \tfrac{1}{4}\overline{\Psi}(\gamma^\alpha\sigma^{\beta\gamma} + \sigma^{\beta\gamma}\gamma^\alpha)\Psi \tag{2.17}$$

where $\sigma^{\alpha\beta} = (i/2)(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)$. Using the identity [10]

$$\gamma^\alpha\sigma^{\beta\gamma} + \sigma^{\beta\gamma}\gamma^\alpha = -2\,\varepsilon^{\alpha\beta\gamma\delta}\gamma_\delta\gamma^5 \tag{2.18}$$

together with formulas (2.1), (2.6), (2.7), (2.9), and (2.10), we reduce formula (2.17) to

$$\begin{aligned} S^{\alpha\beta\gamma} &= -\tfrac{1}{2}\,\varepsilon^{\alpha\beta\gamma\delta}\,\overline{\Psi}\gamma_\delta\gamma^5\Psi = \tfrac{1}{2}\,\varepsilon^{\alpha\beta\gamma\delta}\,J^0_\delta \\ &= 2|\rho|^2\,\varepsilon^{\alpha\beta\gamma\delta}\,A^0_\delta = 2\mathbf{A}^\alpha\cdot\mathbf{A}^\beta\times\mathbf{A}^\gamma \end{aligned} \tag{2.19}$$

The Yang–Mills version of the spin polarization tensor is easily shown from formula (2.14) to be

$$S_g^{\alpha\beta\gamma} = \mathrm{Re}[A_K^{\alpha\beta} A^{K\gamma} - A_K^{\alpha\gamma} A^{K\beta}] \tag{2.20}$$

In the limit of a large coupling constant g, the Yang–Mills formula (2.20) becomes, using the definition of $A^K_{\alpha\beta}$ given in formula (2.15),

$$\lim_{g\to\infty} g^{-1} S_g^{\alpha\beta\gamma} = 2\mathbf{A}^\alpha \cdot \mathbf{A}^\beta \times \mathbf{A}^\gamma \tag{2.21}$$

which equals $S^{\alpha\beta\gamma}$ by formula (2.19). Similarly, we can derive $T^{\alpha\beta}$ and J^α directly from the Yang–Mills formulas.

Both the orthogonal constraint (2.13) and the definition $\mu = (2/g)m$ are used when evaluating the limit in formula (2.16). Quadratic terms in the tensor fields (A^K_α, ρ) which are not multiplied by g in the Yang–Mills Lagrangian (2.14) vanish in the limit (2.16). Quartic terms multiplied by g^2 sum to zero because of the orthogonal constraint (2.13). Thus the limit in formula (2.16) equals the Lagrangian (2.12), which contains only cubic terms in the tensor fields (A^K_α, ρ). These are precisely the cubic terms multiplied by g in the Yang–Mills Lagrangian (2.14).

Note that the existence of the limit in formula (2.16) depends on the cancellation of all quartic terms in the tensor fields (A^K_α, ρ) in the constrained Yang–Mills Lagrangian (2.14) since by formulas (2.14) and (2.16) and the fact that $\mu = (2/g)m$ precisely these quartic terms are multiplied by g^2. Thus, for general Yang–Mills Lagrangians the limit in formula (2.16) fails to exist. In fact, for general Yang–Mills equations wave packets will not propagate as the Yang–Mills coupling constant g becomes large. Because Yang–Mills equations are nonlinear the mass of each plane wave generally depends on its amplitude, which causes wave packets to lose their elementary character due to velocity splitting, which becomes more severe as g becomes large [18]. Furthermore, in general, unless the amplitude vanishes for each plane wave, the mass of each plane wave becomes infinite in the limit of formula (2.16). However, as previously shown, the Yang–Mills equation derived from the Lagrangian (2.14) with the constraints (2.11) and (2.13) has exact plane wave solutions in one-to-one correspondence with the plane wave solutions of Dirac's bispinor equation [5, 19]. The mass of each plane wave equals m, and hence is independent of amplitude. Wave packets are identical to the wave packets derived from Dirac's equation and do not exhibit velocity splitting [19]. Thus, the Euler–Lagrange equation for the constrained Yang–Mills Lagrangian of the specific form (2.14) has solutions similar to Dirac's bispinor equation, which is a limiting case of it by formula (2.16).

Finally, we now present a simplification in the Yang–Mills Lagrangian (2.14) which removes the quartic potential $\frac{1}{2}g^2|\rho|^4$. Note that the part of the Lagrangian (2.14) for the scalar field ρ is not unique. For example, the coupling of the $U(1)$ gauge potential A^0_α to the scalar field ρ, which we henceforth denote as g_0, need not equal the Yang–Mills self-coupling constant

g in formula (2.15). Thus, we may consider instead of the Lagrangian (2.14) the following Lagrangian without the quartic potential, for which the coupling constants g and g_0 are not necessarily equal:

$$L_{g_0} = -\frac{1}{4g}\,\mathrm{Re}[A^K_{\alpha\beta} A^{\alpha\beta}_K] + \frac{1}{g_0}\,\overline{D_\alpha(\rho + \mu)} D^\alpha(\rho + \mu) \tag{2.22}$$

where

$$D_\alpha(\rho + \mu) = \partial_\alpha \rho + i g_0 A^0_\alpha(\rho + \mu) \tag{2.23}$$

so that g_0 couples the scalars ρ and μ to the gauge potential A^0_α. As before, the Yang–Mills self-coupling is denoted as g, so that $A^0_{\alpha\beta}$ and $\mathbf{A}_{\alpha\beta}$ are defined exactly as in formula (2.15). Using the orthogonal constraint (2.13), and neglecting terms which vanish in the limit (2.16), we find that formula (2.22) equals formula (2.14) multiplied by a constant factor g^{-1} provided that

$$g_0 = \frac{3}{2}\,g, \qquad \mu = \frac{2m}{g_0} \tag{2.24}$$

This selection of the constants g_0 and μ eliminates the quartic potential $\frac{1}{2} g^2|\rho|^4$ in the Lagrangian (2.14). Note that the argument following formula (2.16) concerning the existence of nonlinear plane waves and wave packets also applies to the Lagrangian (2.22), as can be deduced from previous work [19]. We henceforth adopt the simpler Lagrangian (2.22) as the tensor form of Dirac's Lagrangian (2.4). Then from formulas (2.16) and (2.22), Dirac's bispinor Lagrangian (2.4) equals

$$L = \lim_{g_0 \to \infty} L_{g_0} \tag{2.25}$$

Note that the Lagrangian (2.22) can be derived from a tetrad Kaluza–Klein model [1, 2], which explicates not only the orthogonal constraint (2.13), but also the limit (2.25). In the tetrad Kaluza–Klein model there are three fundamental constants m, δ, and κ, where m is the fermion mass, δ is a length which characterizes the size of the higher dimensions, and κ is Newton's constant. The constants g, g_0, and μ are functions of κ, δ, and m. In particular, the limit (2.25) is equivalent to the limit where the length δ becomes vanishingly small. It can be shown that δ (denoted as λ^{-1} in previous work [1, 2]) equals the Planck length $\kappa^{1/2}$ divided by $g^{3/2}$ and thus, in the limit required to obtain Dirac's equation, δ is much smaller than the Planck length. Hence the limit (2.25) has a geometric significance in the tetrad Kaluza–Klein model.

3. FIERZ IDENTITIES FOR SPINORS

Fierz identities are the bridge between bispinors and the constrained tensors representing them [5, 16]. In this section we review the derivation of the spinor Fierz identity (3.13), which we generalize in Section 4 to multiplets of $2n$ spinors. We also review the discrete transformation T defined on spinors by formula (3.16), which (for a Minkowski space-time) is associated with reflection through the origin $x^\alpha \to x_\alpha = g_{\alpha\beta} x^\beta$, showing in formula (3.18) that the spinor transformation T defines the Minkowski metric $g_{\alpha\beta}$. Similarly in Section 4 the parity map P acting on multiplets of $2n$ spinors defines the gauge metric g_{JK} used to lower and raise gauge indices in the Yang–Mills Lagrangian (2.22) and in the orthogonal constraint (2.13).

In previous work we called T a "parity map" for spinors and denoted it as P, which we also used to denote the usual parity map for bispinors, which interchanges the left and right component spinors [5, 9]. However, to avoid confusion in this paper, we will use different notation for the discrete transformation T acting on single spinors and the usual parity map P defined for bispinors. In Section 4 we will see that the bispinor parity map P consists of a gauge transformation Ω composed with T.

A spinor is a two-dimensional complex vector, denoted as

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \in C^2 \tag{3.1}$$

Acting on spinors ξ are the 2×2 complex Pauli matrices $\sigma^\alpha = (\sigma^0, \sigma^1, \sigma^2, \sigma^3)$ defined by

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$\sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3.2}$$

We define $\sigma_\alpha = (\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3)$ and denote $\tilde{\sigma}^\alpha = \sigma_\alpha$ and $\tilde{\sigma}_\alpha = \sigma^\alpha$. A straightforward evaluation of the Pauli matrices gives the following trace formula:

$$\mathrm{Tr}[\sigma_\alpha\, \tilde{\sigma}_\beta] = 2g_{\alpha\beta} \tag{3.3}$$

where $g_{\alpha\beta}$ denotes the Minkowski metric tensor (with diagonal elements $\{1, -1, -1, -1\}$ and zeros off the diagonal). A further trace formula is expressed by

$$\mathrm{Tr}[\sigma_\alpha\, \tilde{\sigma}_\delta\, \sigma_\beta\, \tilde{\sigma}_\gamma] = 2C_{\alpha\beta\gamma\delta} \tag{3.4}$$

where, as will be seen in formula (3.13), $C_{\alpha\beta\gamma\delta}$ is a Lorentz tensor. Such a tensor is a linear combination of $g_{\alpha\beta}g_{\gamma\delta}$, $g_{\alpha\gamma}g_{\beta\delta}$, $g_{\alpha\delta}g_{\beta\gamma}$, and $\varepsilon_{\alpha\beta\gamma\delta}$, where $\varepsilon_{\alpha\beta\gamma\delta}$ is the permutation tensor. A straightforward derivation shows that

$$C_{\alpha\beta\gamma\delta} = g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\beta}g_{\gamma\delta} - i\varepsilon_{\alpha\beta\gamma\delta} \tag{3.5}$$

The tensor $C_{\alpha\beta\gamma\delta}$ satisfies numerous identities, chief of which are the symmetries

$$C_{\alpha\beta\gamma\delta} = C_{\beta\alpha\delta\gamma} = C_{\gamma\delta\alpha\beta} = C_{\delta\gamma\beta\alpha} \tag{3.6}$$

and the inversion formula

$$C_{\alpha\beta\gamma\delta}C^{\gamma\delta\lambda\mu} = 4\delta^{\lambda}_{\alpha}\delta^{\mu}_{\beta} \tag{3.7}$$

where δ^{β}_{α} equals one if $\alpha = \beta$ and zero otherwise. Note that the tensor indices α, β, γ, δ, λ, μ are lowered and raised using the Minkowski metric tensor $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$, and all repeated indices are to be summed from 0 to 3.

If we set $\delta = 0$ in formula (3.4), noting that $\tilde{\sigma}_0 = \sigma^0 = I$, where I is the 2×2 identity matrix, we have

$$\mathrm{Tr}[\sigma_\alpha\ \sigma_\beta\ \tilde{\sigma}_\gamma] = 2C_{\alpha\beta\gamma 0} \tag{3.8}$$

Since the Pauli matrices σ_γ are a basis for 2×2 complex matrices, the product $\sigma_\alpha\ \sigma_\beta$ is a linear combination of the matrices σ_γ. From formulas (3.3) and (3.8), this linear combination is given by

$$\sigma_\alpha\sigma_\beta = C^{\gamma 0}_{\alpha\beta}\sigma_\gamma \tag{3.9}$$

By a similar argument, setting $\alpha = 0$ in formula (3.4) gives

$$\tilde{\sigma}_\gamma\tilde{\sigma}_\delta = C^{0\beta}_{\gamma\delta}\tilde{\sigma}_\beta \tag{3.10}$$

Next we consider a pair of spinors ξ and η to which we associate a complex Lorentz four-vector j^α, whose components are defined by the 2×2 matrix

$$j = 2\eta\xi^+ = \begin{bmatrix} j^0 + j^3 & j^1 - ij^2 \\ j^1 + ij^2 & j^0 - j^3 \end{bmatrix} \tag{3.11}$$

where $\xi^+ = (\bar{\xi}_1, \bar{\xi}_2)$ denotes the transpose conjugate of ξ. (The bar denotes ordinary complex conjugation.) The spin group of 2×2 complex matrices with determinant one, denoted $SL(2, C)$ or $Spin(1, 3)$, acts on the spinors ξ and η. Acting on $\eta\xi^+$ in formula (3.11), the spin group leaves invariant the determinant of j, and hence the Minkowski norm of j^α. Thus j^α becomes a Lorentz four-vector.

We can solve for j^α in formula (3.11) by first noting that $j = j^\beta\tilde{\sigma}_\beta$, multiplying by σ_α, and then using the trace formula (3.3). This defines a map j_α: $C^2 \times C^2 \rightarrow C^4$ mapping each pair of spinors ξ and η to a complex Lorentz four-vector $j_\alpha(\xi, \eta)$ given by

$$j_\alpha(\xi, \eta) = \xi^+\sigma_\alpha\eta \tag{3.12}$$

We now derive the following Fierz identity.

Proposition 1. For all $\xi, \eta, \kappa, \nu \in C^2$,

$$2j_\alpha(\xi, \eta)j_\beta(\kappa, \nu) = C^{\gamma\delta}_{\alpha\beta}j_\gamma(\xi, \nu)j_\delta(\kappa, \eta) \tag{3.13}$$

(Note that since j_α, j_β, j_γ, and j_δ are Lorentz four-vectors, $C_{\alpha\beta\gamma\delta}$ is a tensor.)

Proof. From formula (3.11) we have

$$2\eta\xi^+ = j^\alpha(\xi, \eta)\tilde{\sigma}_\alpha \tag{3.14}$$

Then the trace formula (3.4) gives

$$\begin{aligned} 2j_\alpha(\xi, \eta)j_\beta(\kappa, \nu) &= 2(\xi^+\sigma_\alpha\eta)(\kappa^+\sigma_\beta\nu) \\ &= 2\,\mathrm{Tr}[\sigma_\alpha(\eta\kappa^+)\,\sigma_\beta(\nu\xi^+)] \\ &= 1/2\,\mathrm{Tr}[\sigma_\alpha\,\tilde{\sigma}_\delta\,\sigma_\beta\,\tilde{\sigma}_\gamma]\,j^\gamma(\xi, \nu)\,j^\delta(\kappa, \eta) \\ &= C_{\alpha\beta\gamma\delta}\,j^\gamma(\xi, \nu)\,j^\delta(\kappa, \eta) \end{aligned} \tag{3.15}$$

which proves formula (3.13). QED

A discrete transformation T: $C^2 \to C^2$ sends a spinor ξ, as defined in formula (3.1), to its dual conjugate $\tilde{\bar{\xi}}$:

$$T\xi = \tilde{\bar{\xi}} = \begin{bmatrix} \bar{\xi}_2 \\ -\bar{\xi}_1 \end{bmatrix} \tag{3.16}$$

Formula (3.2) gives $T\sigma_\alpha\xi = \tilde{\sigma}_\alpha T\xi$. Hence, $\tilde{\bar{\xi}} = T\xi$ transforms under the conjugate representation of the spin group $SL(2, C)$. Since $T^2\xi = -\xi$, the transformation T is a bijection. From formulas (3.11) and (3.12) we have

$$j^\alpha(T\xi, T\eta) = j_\alpha(\eta, \xi) = \overline{j_\alpha(\xi, \eta)} \tag{3.17}$$

Setting $\eta = \xi$ in formula (3.12), we see that $j^\alpha = j^\alpha(\xi, \xi)$ is real. By formula (3.17), the action of T on real Lorentz four-vectors j^α becomes

$$j^\alpha \xrightarrow{T} j_\alpha = g_{\alpha\beta}j^\beta \tag{3.18}$$

That is, the discrete transformation T reverses the space components of real Lorentz four-vectors j^α. This motivates calling T a "parity map" which acts on spinors $\xi \in C^2$.

The spinor "parity map" T satisfies the following relations, which we will use in Section 4. By formula (3.16), we have for all $\xi, \eta \in C^2$

$$\begin{aligned} T^2\xi &= -\xi \\ (T\xi)^+(T\eta) &= \eta^+\xi \\ (T\xi)^+\eta &= -(T\eta)^+\xi \\ \xi^+(T\eta) &= -\eta^+(T\xi) \end{aligned} \tag{3.19}$$

Also, by formulas (3.2) and (3.16) the spinor parity map T lowers and raises the space-time index of the Pauli matrices on commutation as follows:

$$T\sigma^\alpha = \sigma_\alpha T \tag{3.20}$$

Note that a "parity operation" can be defined for spinor fields $\xi(x^\alpha)$, which combines T with the space reflection sending the space-time point $x^\alpha \in R^4$ to x_α. However, as defined here, T: $C^2 \to C^2$ transforms only the spinor components $\xi \in C^2$.

With the parity map T we can construct the following map s: $C^2 \times C^2 \to C$ which associates a Lorentz scalar $s(\xi, \eta)$ to each pair of spinors ξ and η:

$$s(\xi, \eta) = \xi^+ T\eta \tag{3.21}$$

That $s(\xi, \eta)$ is a Lorentz scalar follows from the fact that the inner product of a spinor ξ and a dual spinor $T\eta$ is invariant for all $SL(2, C)$ transformations. Note from formulas (3.12) and (3.21) that $s(\xi, \eta) = j_0(\xi, T\eta)$, and hence from formulas (3.17) and (3.19) we derive

$$\begin{aligned} s(\xi, \eta) &= -s(\eta, \xi) \\ s(T\xi, T\eta) &= \overline{s(\xi, \eta)} \end{aligned} \tag{3.22}$$

for all spinors $\xi, \eta \in C^2$.

4. FIERZ IDENTITIES FOR MULTIPLETS OF 2*n* SPINORS

In this secton we extend the Fierz identity (3.13) for single spinors to multiplets of $2n$ spinors. Using this extended Fierz identity (4.17), we then derive the generalization of the orthogonal constraint (2.7) for $2n$ spinor multiplets [see formula (4.28)]. Generalization of the orthogonal constraint requires extending the usual parity map P which is defined for bispinors [or spinor doublets as in formula (4.6)] to act on multiplets of $2n$ spinors. We show that the extended parity maps correspond to the unitary symplectic forms on a $2n$-dimensional complex vector space, and hence form a manifold of $n(2n - 1)$ real dimensions. The choice of parity map dictates which gauge generators are associated with interactions that conserve parity. As discussed in the introduction, to conform to observation we must choose parity maps which conserve parity for, and hence commute with, electromagnetic interactions. This condition is not satisfied for the spinor parity map T acting, as in formula (3,16), on each component spinor of a multiplet, whereas the usual parity map P defined for bispinors commutes with electromagnetic gauge transformations.

We will see also in formula (4.11) that the choice of parity map affects the gauge metric used to lower and raise gauge indices in the Yang–Mills Lagrangian (2.22) and in the orthogonal constraint (2.13). We will further discuss the parity map in Section 5 when we consider the electroweak gauge transformations of the Standard Model.

A multiplet of $2n$ spinors is a $4n$-dimensional complex vector, denoted as

$$\xi = \begin{bmatrix} \xi^{(1)} \\ \cdot \\ \cdot \\ \xi^{(2n)} \end{bmatrix} \in C^{4n} \tag{4.1}$$

where $\xi^{(1)}, \ldots, \xi^{(2n)} \in C^2$ are spinors as defined in formula (3.1). Acting on multiplets of $2n$ spinors ξ are the Pauli matrices σ^α, which extend to $4n \times 4n$ matrices by the usual direct sum representation. Also acting on ξ are $4n \times 4n$ gauge matrices t^K, where $K = 0, 1, \ldots, 4n^2 - 1$, which commute with the Pauli matrices σ^α. The gauge matrix $t^0 = I$ is the identity matrix, which is the Hermitian generator of $U(1)$ gauge transformations acting on ξ. The gauge matrices t^K with $K > 0$ are the Hermitian generators of $SU(2n)$ gauge transformations acting on ξ.

Since $\xi \in C^{4n}$, we have the trace of t^K given by

$$\mathrm{Tr}[t^K] = 4n\delta^{0K} \tag{4.2}$$

where δ^{JK} equals 1 if $J = K$ and equals 0 otherwise. That is, for $K = 0$, the trace of the $U(1)$ gauge matrix $t^0 = I$ equals $4n$, whereas, the $SU(2n)$ gauge matrices t^K with $K > 0$ have zero trace. Since the trace can be used to define a positive-definite inner product on the linear space of Hermitian matrices [20], we can choose the Hermitian gauge matrices t^K so that the trace formula (4.2) extends as follows:

$$\mathrm{Tr}[t^J\, t^K] = 4n\delta^{JK} \tag{4.3}$$

We shall refer to a matrix which commutes with the Pauli matrices σ^α as a gauge matrix. Let Ω be a $4n \times 4n$ gauge matrix which is both unitary and skew-symmetric, that is,

$$\begin{aligned} \Omega^+ &= \Omega^{-1} \\ \Omega^T &= -\Omega \end{aligned} \tag{4.4}$$

$$\overline{\Omega} = -\Omega^{-1}$$

where Ω^T, $\overline{\Omega}$, Ω^+, and Ω^{-1} denote the transpose, complex conjugate, transpose complex conjugate, and inverse, respectively, of the gauge matrix Ω. Note that the last formula in (4.4) is a consequence of the first two, which state that Ω is unitary and skew-symmetric. By formula (4.4) the gauge matrix Ω can be regarded as a unitary symplectic form defined on multiplets of $2n$ spinors ξ.

Each Ω can be composed with the parity map T defined in formula (3.16) for single spinors, to obtain a parity map $P = \Omega T$ acting on multiplets of $2n$ spinors ξ as follows:

$$P\xi = \Omega T\xi = \Omega \begin{bmatrix} T\xi^{(1)} \\ \cdot \\ \cdot \\ T\xi^{(2n)} \end{bmatrix} \tag{4.5}$$

where T acts on $\xi \in C^{4n}$ by the usual direct sum representation (i.e., T acts on each spinor component of ξ). Note from formula (3.18) that the parity map T transforms j^α into j_α. We will see a similar result in formula (4.27) for the parity map P.

Since the unitary gauge matrices $U(2n)$ form a Lie group of dimension $4n^2$ and the unitary gauge matrices which leave the symplectic form Ω invariant form a subgroup $Sp_\Omega(n)$ of dimension $n(2n + 1)$, the homogeneous space $U(2n)/Sp_\Omega(n)$ of unitary symplectic forms Ω has dimension $n(2n - 1) = 4n^2 - n(2n + 1)$. Thus, the parity maps P defined in formula (4.5) form a manifold also of dimension $n(2n - 1)$.

For example, for spinor doublets ($n = 1$), the parity maps P form a one-dimensional manifold. In previous work [5] we defined a parity map P on spinor doublets as follows:

$$P\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} T\eta \\ -T\xi \end{bmatrix} \tag{4.6}$$

where $\xi, \eta \in C^2$ are spinors. Other possible definitions of a parity map for spinor doublets differ from this choice of P by a $U(1)$ phase. Mapping spinor doublets to bispinors, as discussed in Section 5, formula (4.6) becomes the usual parity map P acting on bispinors.

By formula (3.19), each spinor component $\xi^{(i)} \in C^2$ of $\xi \in C^{4n}$ for $1 \leq i \leq 2n$ in formula (4.5) satisfies $T^2\xi^{(i)} = -\xi^{(i)}$. We have from formulas (3.16) and (4.5) that $P\Omega = \overline{\Omega}P$. Hence, using formula (4.4) together with (3.19) and (4.5), we get for all $\xi, \eta \in C^{4n}$

$$
\begin{aligned}
P^2\xi &= \xi \\
(P\xi)^+ (P\eta) &= \eta^+\xi \\
(P\xi)^+\eta &= (P\eta)^+\xi \\
\xi^+(P\eta) &= \eta^+(P\xi)
\end{aligned}
\tag{4.7}
$$

Note that from the first formula in (4.7), $P = P^{-1}$ for the parity map P acting on $\xi \in C^{4n}$.

The Hermitian generators t^K of the gauge group $U(2n)$ acting on C^{4n} satisfy trace formulas similar to the trace formulas (3.3) and (3.4) for the Pauli matrices σ^α. Let t be any gauge matrix acting on C^{4n}. That is, t commutes with the Pauli matrices σ^α, but t is not necessarily Hermitian. Define

$$
\tilde{t} = \Omega t^T \Omega^{-1} \tag{4.8}
$$

From formulas (3.16), (4.4), and (4.5), when t is Hermitian (i.e., $t^+ = t$ or equivalently $\bar{t} = t^T$), we derive $\tilde{t} = PtP$, and furthermore $\tilde{t}$ is also Hermitian. Let us denote $t_K = \tilde{t}^K$. Since $P = P^{-1}$, we also have $t^K = P\tilde{t}^K P = Pt_K P = \tilde{t}_K$. Again from formulas (3.20), (4.4), and (4.5) we have

$$
\begin{aligned}
P\sigma^\alpha &= \sigma_\alpha P \\
Pt^K &= t_K P
\end{aligned}
\tag{4.9}
$$

Formula (4.3) can be written as

$$
\mathrm{Tr}[t^J\, \tilde{t}_K] = 4n\, \delta^J_K \tag{4.10}
$$

For $4n \times 4n$ matrices σ_α and t_K, we have, similar to formula (3.3),

$$
\begin{aligned}
\mathrm{Tr}[\sigma_\alpha\, \tilde{\sigma}_\beta] &= 4n g_{\alpha\beta} \\
\mathrm{Tr}[t_J\, \tilde{t}_K] &= 4n g_{JK}
\end{aligned}
\tag{4.11}
$$

which defines the gauge metric g_{JK}. Using formulas (4.4) and (4.8) and the invariance of trace for cyclically permuting and transposing matrices, we see that g_{JK} is symmetric and real ($g_{JK} = g_{KJ}$ and $\bar{g}_{JK} = g_{JK}$). If we define a set of coefficients g'_{JK} such that $t_J = g'_{JK}\, t^K$, then substitution into formula (4.11) using (4.10) shows that $g'_{JK} = g_{JK}$. Thus, $t_J = g_{JK}\, t^K$. Similarly, from formula (4.8) we have $\tilde{t}_J = g_{JK}\tilde{t}^K$. Moreover, $g_{JK}\tilde{t}^K = \tilde{t}_J = t^J$ shows that g_{JK} has full rank, and thus is nondegenerate. Hence, gauge indices J, K, L are lowered and raised using the gauge metric g_{JK} and its inverse g^{JK}.

Likewise, formula (3.4) becomes

$$
\begin{aligned}
\mathrm{Tr}[\sigma_\alpha\, \tilde{\sigma}_\delta\, \sigma_\beta\, \tilde{\sigma}_\gamma] &= 4nC_{\alpha\beta\gamma\delta} \\
\mathrm{Tr}[t_J\, \tilde{t}_M\, t_K\, \tilde{t}_L] &= 4\, nC_{JKLM}
\end{aligned}
\tag{4.12}
$$

which defines the tensor C_{JKLM}.

The $4n \times 4n$ matrices σ_α and t_K commute for each index α and K. Moreover,

$$\mathrm{Tr}[\sigma_\alpha]\ \mathrm{Tr}[t_K] = 4n\ \mathrm{Tr}[\sigma_\alpha t_K] = 16n^2\delta^0_\alpha\delta^0_K \tag{4.13}$$

As with σ_α in formulas (3.9) and (3.10), products of the gauge matrices t_K (or $\tilde{t}_K$) can be expressed as linear combinations of the t_K (or $\tilde{t}_K$), e.g.,

$$\tilde{t}_L\tilde{t}_M = C^{0K}_{LM}\tilde{t}_K \tag{4.14}$$

From this observation and formula (4.13), if σ is a product of Pauli matrices σ_α and t is a product of gauge matrices t_K, then

$$\mathrm{Tr}[\sigma]\ \mathrm{Tr}[t] = 4n\ \mathrm{Tr}[\sigma t] \tag{4.15}$$

Now consider the map j^K_α: $C^{4n} \times C^{4n} \to C^{16n^2}$ mapping each pair of spinor multiplets ξ, $\eta \in C^{4n}$ to a set of $4n^2$ complex Lorentz four-vectors $j^K_\beta(\xi, \eta)$, defined by

$$j^K_\beta(\xi, \eta) = \xi^+\sigma_\beta\, t^K\eta \tag{4.16}$$

where ξ^+ denotes the transpose conjugate of the spinor multiplet ξ. We will derive the following Fierz identity for spinor multiplets.

Proposition 2. For all ξ, η, κ, $\nu \in C^{4n}$,

$$C^{LM}_{JK}j^J_\alpha(\xi, \eta)j^K_\beta(\kappa, \nu) = nC^{\gamma\delta}_{\alpha\beta}j^L_\gamma(\xi, \nu)j^M_\delta(\kappa, \eta) \tag{4.17}$$

Proof. The proof is similar to the proof of Proposition 1 in Section 3, using the definition (4.16) with the trace formulas (4.11), (4.12), and (4.15). In particular, similar to formula (3.14), we derive

$$(4n)\eta\xi^+ = j^{\beta K}(\xi, \eta)\tilde{\sigma}_\beta\tilde{t}_K \tag{4.18}$$

An equation similar to formula (3.15) is then obtained. The inversion formula (3.7) is used in the final step. QED

Note that since $\sigma^\beta t^K = \tilde{\sigma}_\beta\tilde{t}_K$ form a basis of $4n \times 4n$ matrices, formulas (4.16) and (4.18) are equivalent ways, via the trace formulas (4.11) and (4.15), to define the Lorentz four-vectors $j^K_\beta(\xi, \eta)$.

Applications to date use a reduced form of the Fierz identity (4.17), which exploits an associative binary operation, denoted as $\otimes$, defined on C^{4n^2} as follows. Let a^K, b^K, and c^K be complex numbers indexed by the set of gauge indices $K = 0, 1, \ldots, 4n^2 - 1$. Define $c = a \otimes b$ if and only if

$$c^K = C^{0K}_{LM}\, a^Lb^M \tag{4.19}$$

Since from formula (4.14), when $c = a \otimes b$, we have

$$(a^L \tilde{t}_L)(b^M \tilde{t}_M) = c^K \tilde{t}_K \tag{4.20}$$

we see that the binary operation $\otimes$ is associative. Moreover, since $t^K = \tilde{t}_K$ was defined [see formula (4.3)] independently of the parity map (4.5), the associative operation $\otimes$ is therefore independent of which parity map is chosen.

Setting $L = 0$ in the Fierz identity (4.17), we get from formula (4.19)

$$[j_\alpha(\xi, \eta) \otimes j_\beta(\kappa, \nu)]^K = nC^{\gamma\delta}_{\alpha\beta} j^0_\gamma(\xi, \nu) j^K_\delta(\kappa, \eta) \tag{4.21}$$

where the Kth component of $c = a \otimes b$ is denoted as $c^K = [a \otimes b]^K$. The reduced Fierz identity (4.21) is independent of the choice of parity map (4.5).

From formulas (4.7), (4.9), and (4.16), the parity map P: $C^{4n} \to C^{4n}$ defined by formula (4.5) transforms the Lorentz four vectors $j^K_\alpha(\xi, \eta)$ as follows:

$$j^K_\alpha(P\xi, P\,\eta) = j^\alpha_K(\eta, \xi) = \overline{j^\alpha_K(\xi, \eta)} \tag{4.22}$$

Similar to formula (3.21), we define a map s: $C^{4n} \times C^{4n} \to C$ mapping each pair of spinor multiplets ξ and η to a complex Lorentz scalar $s(\xi, \eta)$ given by

$$s(\xi, \eta) = \xi^+ P\eta \tag{4.23}$$

That $s(\xi, \eta)$ is a Lorentz scalar follows from the spinor map (3.21) and the fact that $P = \Omega T$. Note from formulas (4.16) and (4.23) that $s(\xi, \eta) = j^0_0(\xi, P\eta)$, and hence from formulas (4.7) and (4.22) we derive

$$\begin{aligned} s(\xi, \eta) &= s(\eta, \xi) \\ s(P\xi, P\eta) &= \overline{s(\xi, \eta)} \end{aligned} \tag{4.24}$$

for all multiplets of $2n$ spinors $\xi, \eta \in C^{4n}$.

Let us now define a map $\xi \to (j^K_\alpha, s)$ taking each multiplet of $2n$ spinors ξ to a set of real Lorentz four-vectors j^K_α for $K = 0, 1, \ldots, 4n^2 - 1$ and a complex Lorentz scalar s given as follows:

$$\begin{aligned} j^K_\alpha &= j^K_\alpha(\xi, \xi) \\ s &= s(\xi, \xi) \end{aligned} \tag{4.25}$$

Setting $\eta = \xi$ in formulas (4.16) and (4.23), we see that the Lorentz four-vectors j^K_α are real, whereas the Lorentz scalar s is complex. Note that apart from the exceptional set where $s = 0$, the map $\xi \to (j^K_\alpha, s)$ is a double covering map (onto its image). Formulas (4.22) and (4.24) give

$$\begin{aligned} j^K_\alpha(P\xi, P\xi) &= j^\alpha_K(\xi, \xi) \\ s(P\xi, P\xi) &= \overline{s(\xi,\xi)} \end{aligned} \tag{4.26}$$

so by formula (4.25), the parity map P transforms j^K_α and s as follows:

$$j_\alpha^K \xrightarrow{P} j_K^\alpha = g^{\alpha\beta} g_{KL} j_\beta^L$$

$$s \xrightarrow{P} \bar{s} \tag{4.27}$$

The following proposition uses the parity map P together with the Fierz identity (4.21) to show that j_α^K and s satisfy an orthogonal constraint.

Proposition 3. The following orthogonal constraint is satisfied by the j_α^K and s in the image of the map $\xi \rightarrow (j_\alpha^K, s)$:

$$j_\alpha^K j_{K\beta} = n|s|^2 g_{\alpha\beta} \tag{4.28}$$

Proof. We will first show that

$$\begin{aligned} j_\alpha^0(\xi, P\xi) &= s\delta_\alpha^0 \\ j_\alpha^0(P\xi, \xi) &= \bar{s}\delta_\alpha^0 \end{aligned} \tag{4.29}$$

By formula (4.22), the second equation follows from the first, so it suffices to prove the first equation of formula (4.29). Using formula (4.16), let us define

$$s_\alpha = \xi^+ \sigma_\alpha P\xi = j_\alpha^0(\xi, P\xi) \tag{4.30}$$

Since $s = s_0$ by formulas (4.23) and (4.25), it suffices to show that $s_\alpha = 0$ for $\alpha = 1, 2, 3$. From formulas (4.7) and (4.9) we have

$$s_\alpha = \xi^+ \sigma_\alpha P\xi = (\sigma_\alpha \xi)^+ P\xi = \xi^+ P\sigma_\alpha \xi = \xi^+ \sigma^\alpha P\xi = s^\alpha \tag{4.31}$$

That is, $s^\alpha = s_\alpha$. Since the index α is lowered by the Minkowski metric, s_0 is the only nonvanishing component of s_α. Thus formula (4.29) is proved.

Since by formula (4.29), s and $\bar{s}$ are the only nonvanishing components of j_α^0 $(\xi, P\xi)$ and $j_\alpha^0(P\xi, \xi)$, on substituting ξ, ξ, $P\xi$, $P\xi$ for ξ, η, κ, ν, respectively, in the Fierz identity (4.21) and setting $K = 0$, we have, using formulas (4.19) and (4.22),

$$C_{LM}^{00} j_\alpha^L j_M^\beta = n|s|^2 C_{\alpha\beta}^{00} \tag{4.32}$$

From formulas (4.3), (4.11), and (4.12) we derive

$$\begin{aligned} C_{\alpha\beta}^{00} &= C_{00\alpha\beta} = \delta_\alpha^\beta \\ C_{LM}^{00} &= C_{00LM} = \delta_L^M \end{aligned} \tag{4.33}$$

Substituting formula (4.33) into (4.32) gives (4.28). QED

In previous work [5], it was shown that each factorization of the second-order Klein–Gordon wave equation into a first-order Dirac equation determined a flavor parameter $c \in R^{4n^2}$ satisfying $\tilde{c} \otimes c = (-1, 0, \ldots, 0)$, where

$\tilde{c}^K = c_K$. [For notation see formulas (4.19)–(4.21).] Defining the flavor map t: $C^{4n} \to C^{4n}$ by $t = -c^K t_K$, we have from formula (4.20) that $\tilde{c} \otimes c = (-1, 0, \ldots, 0)$ if and only if $t\tilde{t} = -t^0 = -I$. For the case $n = 1$, the usual parity map P is defined on spinor doublets by formula (4.6), and the flavor map t by

$$t \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \xi \\ -\eta \end{bmatrix} \tag{4.34}$$

where $\xi, \eta \in C^2$ are spinors. For $n > 1$, the parity map P and the flavor map t are extended in standard models to act on multiplets of $2n$ spinors by the usual direct sum representation. Proposition 4 which follows generalizes Proposition 3 using the flavor parameter c^K, and Proposition 5 extends the parity map.

Proposition 4. Let the Lorentz four-vectors j^K_α for $K = 0, 1, \ldots, 4n^2 - 1$ and scalar s be defined as in Proposition 3. Let $c \in R^{4n^2}$ satisfying $\tilde{c} \otimes c = (-1, 0, \ldots, 0)$ be a flavor parameter. Define

$$J^K_\alpha = [j_\alpha \otimes \tilde{c}]^K \tag{4.35}$$

Then J^K_α and s satisfy the orthogonal constraint

$$J^K_\alpha J_{K\beta} = -n|s|^2 g_{\alpha\beta} \tag{4.36}$$

Proof. Using formulas (4.19) and (4.33), we can express formula (4.32) as

$$[j_\alpha \otimes \tilde{j}_\beta]^0 = n|s|^2 g_{\alpha\beta} \tag{4.37}$$

By formula (4.8) we have $(t_J t_K)\tilde{} = \tilde{t}_K \tilde{t}_J$, and therefore $c = a \otimes b$ in formula (4.20) implies that $\tilde{c} = \tilde{b} \otimes \tilde{c}$. Let $c \in R^{4n^2}$ be a flavor parameter, which satisfies $\tilde{c} \otimes c = (-1, 0, \ldots, 0)$. Let $J_\alpha = j_\alpha \otimes \tilde{c}$ as in formula (4.35), so that $\tilde{J}_\alpha = c \otimes \tilde{j}_\alpha$. We have from formulas (4.19), (4.33), and (4.37)

$$\begin{aligned} J^K_\alpha J_{K\beta} &= C^{00}_{LM} J^L_\alpha \tilde{J}^M_\beta \\ &= [J_\alpha \otimes \tilde{J}_\beta]^0 = [j_\alpha \otimes \tilde{c} \otimes c \otimes \tilde{j}_\beta]^0 = -[j_\alpha \otimes \tilde{j}_\beta]^0 \\ &= -n|s|^2 g_{\alpha\beta} \end{aligned}$$

which proves formula (4.36). QED

Proposition 5. The parity map P transforms J^K_α and s as follows:

$$J^K_\alpha \xrightarrow{P} \bar{J}^\alpha_K = g^{\alpha\beta} g_{KL} \bar{J}^L_\beta$$

$$s \xrightarrow{P} \bar{s} \tag{4.39}$$

Proof. The parity map P transforms the flavor map $t = -c^K t_K$ into $\tilde{t} =$

$-c^K \tilde{t}_K = -\tilde{c}^K t_K$. Thus, P transforms c into $\tilde{c}$. From formula (4.27), P transforms j_α and s into $\tilde{j}^\alpha$ and $\bar{s}$. Thus, P transforms $J_\alpha = j_\alpha \otimes \tilde{c}$ into $K^\alpha = \tilde{j}^\alpha \otimes c$. For the Hermitian generators t_K we have $(t_J t_K)^+ = t_K t_J$. Therefore, $c = a \otimes b$ in formula (4.20) implies that $\bar{c} = \bar{b} \otimes \bar{a}$. Noting as in the proof of Proposition 4 that $\tilde{J}_\alpha = c \otimes \tilde{j}_\alpha$, and that both c and $\tilde{j}^\alpha$ are real, we see that $\bar{K}^\alpha = c \otimes \tilde{j}^\alpha = \tilde{J}^\alpha$. That is, P transforms J^K_α into $\bar{J}^\alpha_K$, which proves formula (4.39). QED

We end this section by noting an equivalent expression for the Lorentz four-vectors J^K_α defined by formula (4.35) in terms of the flavor map $t = -c^K t_K$. More generally, for $\xi, \eta \in C^{4n}$ define

$$J^K_\alpha(\xi, \eta) = [j_\alpha(\xi, \eta) \otimes \tilde{c}]^K \tag{4.40}$$

A straightforward derivation using formulas (4.14), (4.19), and (4.20) gives

$$J^K_\alpha(\xi, \eta) = -j^K_\alpha(t\xi, \eta) \tag{4.41}$$

for all $\xi, \eta \in C^{4n}$. In particular, when $\xi = \eta$ we can define J^K_α in Proposition 4 by

$$J^K_\alpha = -j^K_\alpha(t\xi, \xi) \tag{4.42}$$

Note that apart from the exceptional set where $s = 0$, the map $\xi \to (J^K_\alpha, s)$ is a double covering map whose image contains only J^K_α which vanish except for an $SU(n, n) \times U(1)$ subgroup of $SL(2n, C) \times U(1)$. Here, $SU(n, n)$ is the subgroup of $SL(2n, C)$ gauge transformations which leave invariant the flavor map t regarded as a Hermitian form. This Hermitian form occurs in the Dirac Lagrangian (5.14) for which J^K_α are the $SL(2n, C) \times U(1)$ Noether currents. The reader may wish to review the case $n = 1$ discussed in detail in Section 2, noting formulas (2.9), (2.11), (4.34), and (4.42), and the isomorphism $SU(1, 1) \cong SL(2, R)$.

5. TENSOR DIRAC THEORY FOR MULTIPLETS OF *n* BISPINOR FIELDS

In this section we generalize the Yang–Mills Lagrangian (2.22) to multiplets of n bispinor fields $\Psi = (\Psi_1, \ldots, \Psi_n)$ in a way that is consistent with standard electroweak theory. We define a map taking Ψ to tensor fields (A^K_α, ρ_j), where A^K_α are $SL(2n, C) \times U(1)$ gauge potentials and ρ_j are n complex scalar fields which the electroweak gauge transformations leave invariant.

In the Standard Model the Higgs field ϕ is subject to a strong quartic potential, causing its equilibrium values to lie in a Goldstone manifold [9]. In formula (5.18) we will see that the Goldstone manifold can be identified with a submanifold of the unitary symplectic forms Ω defined in Section 4.

As in formula (4.5), each Ω defines a parity map $P = \Omega T$, where T is the spinor parity map (3.16). Thus, we will show that a one-to-one correspondence exists between vacuum orientations of the Higgs field (i.e., the equilibrium values of ϕ) and parity maps P. Linking the Higgs field to the parity maps P reveals that the Standard Model Higgs field has a symplectic structure, as described by formula (5.18), which was not previously recognized.

Additional features of the tensor model are also discussed in this section. In the Standard Model, the W and Z boson mass ratio is determined by a fixed rotation of the electroweak gauge generators through the Weinberg angle [9]. In this section we show that a similar process determines the fermion mass ratios in the tensor Dirac theory. Instead of the Weinberg rotation, we show that a fixed self-adjoint transformation of the $SL(2n, C) \times U(1)$ gauge generators, which restricts to a dilation on each flavor Lie subalgebra, produces the mass ratios within a fermion multiplet.

As discussed in Section 1, field quantization requires identifying the physically realizable solutions of the classical field equations. In the Standard Model arbitrary superpositions of flavors within a fermion multiplet (e.g., a superposition of electron and neutrino flavors) are not physically realizable and hence are forbidden [9]. It can be shown that such a superselection rule follows from the tensor theory. Based on the argument in Section 2 preceding formula (2.22) concerning the existence of nonlinear plane waves and wave packets, in Proposition 6 we will show that nonlinear wave packets derived from the Yang–Mills Lagrangian (5.30) with the orthogonal constraint (5.22) are always associated with a single fermion flavor, and hence only these can propagate.

A bispinor $\Psi \in C^4$ consists of a spinor $\xi \in C^2$ and a dual conjugate spinor $\tilde{\eta} = T\eta \in C^2$ as defined in formula (3.16). That is,

$$\Psi = \begin{bmatrix} \xi \\ T\eta \end{bmatrix} \in C^4 \tag{5.1}$$

where $\xi, \eta \in C^2$ are spinors. The charge conjugate of Ψ is defined by

$$\Psi^C = \begin{bmatrix} \eta \\ T\xi \end{bmatrix} \tag{5.2}$$

and the Dirac matrices γ^α for $\alpha = 0, 1, 2, 3$ acting on Ψ are defined by

$$\gamma^\alpha \Psi = \begin{bmatrix} T\sigma^\alpha \eta \\ \sigma^\alpha \xi \end{bmatrix} \tag{5.3}$$

The fifth Dirac matrix γ^5 acts on Ψ by

$$\gamma^5\Psi = \begin{bmatrix} \xi \\ -T\eta \end{bmatrix} \tag{5.4}$$

We denote the bijective map from spinor doublets to bispinors by B: $C^4 \to C^4$. That is, from formula (5.1),

$$\Psi = B\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \xi \\ T\eta \end{bmatrix} \tag{5.5}$$

From formulas (5.3) and (5.5), the parity map P defined for spinor doublets in formula (4.6), acting on bispinors, is give by $\gamma^0 = BPB^{-1}$. Similarly, the flavor map t defined for spinor doublets in formula (4.34), acting on bispinors, is given by $\gamma^5 = BtB^{-1}$.

Let $\Psi = (\Psi_1, \ldots, \Psi_n)$ be a multiplet of n bispinors Ψ_j, where $j = 1, 2, \ldots, n$. We extend formulas (5.1)–(5.5) to Ψ by the usual direct sum representation. In particular, the bijective map (5.5) extends to a bijective map B: $C^{4n} \to C^{4n}$ from multiplets of $2n$ spinors to multiplets of n bispinors as follows:

$$B\begin{bmatrix} \xi^{(1)} \\ \eta^{(1)} \\ \cdot \\ \cdot \\ \xi^{(n)} \\ \eta^{(n)} \end{bmatrix} = \begin{bmatrix} \xi^{(1)} \\ T\eta^{(1)} \\ \cdot \\ \cdot \\ \xi^{(n)} \\ T\eta^{(n)} \end{bmatrix} \tag{5.6}$$

Equivalently, each component bispinor Ψ_j of $\Psi = (\Psi_1, \ldots, \Psi_n)$ is defined by

$$\Psi_j = \begin{bmatrix} \xi^{(j)} \\ T\eta^{(j)} \end{bmatrix} \tag{5.7}$$

where $\xi^{(j)}, \eta^{(j)} \in C^2$ are spinors for $j = 1, 2, \ldots, n$. Charge conjugation, Dirac matrices γ^α, the parity map γ^0, and flavor map γ^5 are defined on the component bispinors Ψ_j as in formulas (5.2)–(5.4).

Standard projections π_j: $C^{4n} \to C^{4n}$ for $1 \leq j \leq n$ map each bispinor multiplet $\Psi = (\Psi_1, \ldots, \Psi_n)$ to a bispinor multiplet $\pi_j\Psi$, all of whose component bispinors vanish, except for the jth, which equals Ψ_j. That is,

$$\pi_j\Psi = (0, \ldots, 0, \Psi_j, 0, \ldots, 0) \tag{5.8}$$

The projections π_j act on multiplets of $2n$ spinors by commutation with the map B: $C^{4n} \to C^{4n}$ defined in formula (5.6).

In all subsequent formulas, repeated indices $j = 1, 2, \ldots, n$ labeling the component bispinors Ψ_j of a bispinor multiplet $\Psi = (\Psi_1, \ldots, \Psi_n,)$ are not subject to the summation convention. For such indices summation will be indicated by a summation symbol $\Sigma_{j=1}^{n}$. The convention of summing over repeated indices will be reserved for space-time indices $\alpha = 0, 1, 2, 3$ and gauge indices $K = 0, 1, \ldots, 4n^2 - 1$.

Note that acting on multiplets of $2n$ spinors, the projections π_j commute with the parity map P and satisfy the following relations:

$$\begin{aligned} \sum_{j=1}^{n} \pi_j &= I \\ \pi_j \pi_k &= \pi_j\, \delta_{jk} \\ \pi_j^+ &= \pi_j \\ \pi_j P &= P\pi_j = \pi_j P \pi_j \\ \pi_j \Omega &= \Omega \pi_j^T = \pi_j \Omega \pi_j^T \end{aligned} \tag{5.9}$$

for all $1 \le j, k \le n$. Moreover, the projections π_j also commute with the Pauli matrices σ^α and the flavor map t.

We define the jth flavor subspace to be the image of the jth projection π_j: $C^{4n} \to C^{4n}$ for each $1 \le j \le n$. We denote the parity map P restricted to the jth flavor subspace by $P_j = \pi_j P$. By formula (5.9), the restricted parity map P_j can be written as $P_j = \Omega_j T$, where $\Omega_j = \pi_j \Omega$ is a (degenerate) symplectic form satisfying

$$\begin{aligned} \Omega_j^T &= -\Omega_j \\ \Omega_j\, \Omega_j^+ &= \pi_j \\ \Omega_j^+\, \Omega_j &= \pi_j^T \end{aligned} \tag{5.10}$$

By the map B: $C^{4n} \to C^{4n}$, every real linear transformation defined for multiplets of $2n$ spinors $\xi \in C^{4n}$ induces a transformation on multiplets of n bispinors $\Psi = B\xi$. Generally, the transformations acting on ξ and $\Psi = B\xi$ are denoted by different symbols as shown in Table I.

Table I. Dictionary of Corresponding Spinor and Bispinor Notation

Transformation	Spinor notation	Bispinor notation
Pauli matrices	σ^δ	$\alpha^\delta = \gamma^0\gamma^\delta$
Parity map	P	$\beta = \gamma^0$
Restricted parity maps	P_j	$\beta_j = \gamma^0\pi_j$
Flavor map	t	γ^5
Flavor generator	it	i
Chiral generator	i	$i\gamma^5$

For example, using formula (3.20), the induced action of the Pauli matrices σ^δ on the bispinor $\Psi \in C^4$ defined in formula (5.1) becomes

$$\alpha^\delta \, \Psi = \begin{bmatrix} \sigma^\delta \xi \\ T\sigma^\delta \eta \end{bmatrix} = \begin{bmatrix} \sigma^\delta \xi \\ \sigma_\delta T \eta \end{bmatrix} \tag{5.11}$$

so that from formula (5.3) the Dirac matrices γ^δ satisfy

$$\gamma^\delta \Psi = \beta \alpha^\delta \Psi \tag{5.12}$$

where $\beta = \gamma^0$ is the bispinor parity map.

Extending formulas (5.11) and (5.12) to bispinor multiplets allows us to write the Dirac Lagrangian as follows:

$$L = \mathrm{Re}[i\Psi^+ \alpha^\delta \, \partial_\delta \Psi - \sum_{j=1}^{n} m_j \Psi^+ \beta_j \Psi] \tag{5.13}$$

where m_j is the jth fermion mass, and $\beta_j = \pi_j \beta$ is the parity map β restricted to the jth flavor (see Table I).

Note that care must be taken with the Lagrangian (5.13) in passing from bispinor multiplets Ψ to $2n$ spinor multiplets ξ using formula (5.6), since the gauge transformations i and $i\gamma^5$ acting on Ψ are induced from the gauge transformations it and i acting on $\xi = B^{-1} \, \Psi$. (See Table I.) Thus, the Dirac Lagrangian (5.13) equals

$$L = \mathrm{Re}[i\xi^+ t\sigma^\delta \, \partial_\delta \xi - \sum_{j=1}^{n} m_j s_j] \tag{5.14}$$

where we define

$$s_j = \xi^+ P_j \xi \tag{5.15}$$

where $P_j = \pi_j P$ and P is the parity map. [See Table I and formulas (4.23), (4.25), and (5.9).]

Our goal is to generalize the Yang–Mills Lagrangian (2.22) to multiplets of n bispinor fields in a manner which is consistent with the standard electroweak model [9]. To construct this generalization we must consider the $SL(2n, C) \times U(1)$ gauge transformations, whose action was described on multiplets of $2n$ spinors in Section 4, induced on multiplets of n bispinors by the map (5.6).

It is straightforward to check that the generators for electroweak $SU(2) \times U(1)$ gauge transformations acting on multiplets of $2n$ spinors have zero trace [9]. Therefore, electroweak gauge transformations form an $SU(2) \times U(1)$ subgroup of $S(U(n) \times U(n)) \subset SU(n, n)$. This result is easily proved using the map B: $C^{4n} \to C^{4n}$ from multiplets of $2n$ spinors to multiplets of n bispinors defined by formula (5.6).

In particular, the chiral $U(1)$ generator i (see Table I) has nonzero trace, so that chiral $U(1)$ gauge transformations are not associated with the electroweak force. Hence, Higgs fields in the Standard Model are invariant scalars for these chiral $U(1)$ gauge transformations, which do not lie in $S(U(n) \times U(n))$. We shall denote the $SU(2) \times U(1)$ electroweak gauge group as G to avoid confusion with other subgroups of the full gauge group $SL(2n, C) \times U(1)$.

A parity map P acting on multiplets of $2n$ spinors ξ as in formula (4.5) can be explicitly defined using the Higgs field $\phi: R^4 \to C^2$ from the standard electroweak model, which has the form [9]

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \tag{5.16}$$

In the Standard Model the Higgs field ϕ is subject to a strong quartic potential, causing the equilibrium values of ϕ to lie in a hypersphere $S^3 \subset C^2$ centered at $0 \in C^2$. By normalizing ϕ, we may assume that S^3 is a unit sphere, so that the Higgs field ϕ at equilibrium satisfies

$$|\phi_1|^2 + |\phi_2|^2 = 1 \tag{5.17}$$

For spinor quadruplets ($n = 2$) in the Standard Model, the unitary symplectic form Ω defining the parity map $P = \Omega T$ in formula (4.5) becomes

$$\Omega = \begin{bmatrix} 0 & \overline{\phi}_2 & 0 & -\overline{\phi}_1 \\ -\overline{\phi}_2 & 0 & -\phi_1 & 0 \\ 0 & \phi_1 & 0 & \phi_2 \\ \overline{\phi}_1 & 0 & -\phi_2 & 0 \end{bmatrix} \tag{5.18}$$

To derive formula (5.18), first note that formula (5.18) agrees with formula (4.6) extended to spinor quadruplets when

$$\phi = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5.19}$$

We then obtain formula (5.18) by applying gauge transformations from G simultaneously to formulas (5.19) and (4.6) extended to spinor quadruplets [noting that $G = SU(2) \times U(1)$], again using the fact that $P = \Omega T$ and that the $SU(2)$ part of G acts on the “left-handed” spinor components.

To make the fermion mass terms invariant under electroweak gauge transformations, the Higgs scalars (5.16) are used as coefficients in the standard electroweak Lagrangian as follows. First note from formula (5.18) that the parity map $P = \Omega T$ depends linearly on the Higgs field ϕ. The same derivation shows that the restricted parity maps $P_j = \Omega_j T$ [see Table I and formula (5.10)] also depend linearly on ϕ. Thus the complex scalars s_j in formula (5.15) depend linearly on ϕ. By construction each scalar field s_j transforms as an invariant scalar under the electroweak gauge group G. This construction makes the standard electroweak Lagrangian (5.14) G-invariant [9].

A straightfoward derivation shows that the J_α^K defined in formula (4.42) are the $SL(2n, C) \times U(1)$ Noether currents obtained from the Lagrangian (5.14). Furthermore, the complex scalar field s defined in formulas (4.23) and (4.25) is from formula (5.15) equal to

$$s = \sum_{j=1}^{n} s_j = \xi^+ P \xi \tag{5.20}$$

Similar to formula (2.10), we map a subset of $SL(2n, C) \times U(1)$ gauge potentials A_α^K and complex scalar fields ρ_j into J_α^K and s_j by setting

$$J_\alpha^K = 4 \sum_{j=1}^{n} |\rho_j|^2 A_\alpha^K$$
$$s_j = \frac{4\left(\sum_{j=1}^{n} |\rho_j|^2\right)^{3/2}}{\left|\sum_{j=1}^{n} \rho_j\right|} \bar{\rho}_j \tag{5.21}$$

Then the orthogonal constraint from formula (4.36) becomes

$$h_{JK} A_\alpha^J A_\beta^K = -\sum_{j=1}^{n} |\rho_j|^2 g_{\alpha\beta} \tag{5.22}$$

where $h_{JK} = n^{-1} g_{JK}$. Apart from the exceptional set where $\rho_j = 0$ (or equivalently $s_j = 0$) for all $1 \leq j \leq n$, the composite map

$$\Psi \xrightarrow{B^{-1}} \xi \rightarrow (J_\alpha^K, s_j) \rightarrow (A_\alpha^K, \rho_j) \tag{5.23}$$

is a well-defined double covering map which is consistent with the orthogonal constraint (5.22) imposed on the tensor fields (A_α^K, ρ_j), and whose image contains only A_α^K which vanish except for an $SU(n, n) \times U(1)$ subgroup of $SL(2n, C) \times U(1)$.

From formulas (4.39) and (5.21), the parity map P acts on the tensor fields (A_α^K, ρ_j) as follows:

$$A_\alpha^K \xrightarrow{P} \bar{A}_K^\alpha = g^{\alpha\beta} g_{KL} \bar{A}_\beta^L$$
$$\rho_j \xrightarrow{P} \bar{\rho}_j \tag{5.24}$$

Fermion masses are generated in the tensor theory by a simple transformation of the $SL(2n, C) \times U(1)$ Hermitian gauge group generators t_K. Recall from formula (2.24) that for a single bispinor field, the fermion mass $m = \frac{1}{2}g_0\mu$ in the Lagrangian (2.22) is proportional to the Yang–Mills coupling constant g_0. Thus, we can change the mass m to λm, where $\lambda > 0$, by mapping each $SL(2, R) \times U(1)$ Hermitian gauge generator t_K to λt_K. This changes the Lie algebra structure constants f^L_{JK} to λf^L_{JK} (hence the Yang–Mills coupling constants g and g_0 become λg and λg_0), without changing the gauge metric g_{JK}.

For the general construction, let the fermion masses in the Lagrangian (5.14) be expressed in terms of a single mass m by $m_j = \lambda_j m$, where $\lambda_j > 0$ for $1 \leq j \leq n$. We define a map Λ: $C^{4n} \to C^{4n}$ by

$$\Lambda = \sum_{j=1}^{n} \sqrt{\lambda_j}\, \pi_j \tag{5.25}$$

As with the projections π_j, the map Λ commutes with the parity map P and the flavor map t as well as the bispinor map B.

Instead of the map (5.23), we consider the map

$$\Psi \overset{\Lambda^{-1}}{\to} \Psi' \overset{B^{-1}}{\to} \xi \to (J^K_\alpha, s_j) \to (A^K_\alpha, \rho_j) \tag{5.26}$$

Note that the map (5.26) has no effect on the orthogonal constraint (5.22). By the map $B^{-1} \circ \Lambda^{-1}$, the Lagrangian (5.13) becomes

$$L = \sum_{j=1}^{n} \mathrm{Re}[i\lambda_j \xi^+ t_j \sigma^\delta\, \partial_\delta\, \xi - m\lambda_j^2 s_j] \tag{5.27}$$

where $t_j = \pi_j t$, and, as previously defined, $m_j = \lambda_j m$ and $s_j = \xi^+ P_j \xi$.

Before applying the transformation Λ to the Hermitian $SL(2n, C) \times U(1)$ gauge generators t_K, we first make a preliminary transformation of the t_K, to account for the unequal coupling constants g_0 and g. By a well-known argument, the unequal coupling constants are accounted for in Yang–Mills theories by noting that the generators in the center of a Lie algebra can be multiplied by the scalar g_0/g without affecting the commutation relations [9]. We define the jth flavor subgroup, denoted as G_j, of the gauge group $G = SL(2n, C) \times U(1)$, to be the subgroup whose elements act as the identity on all flavor subspaces except for the jth flavor subspace. Let Z be the centralizer of the subgroup generated by the G_j for $1 \leq j \leq n$. That is, each element of Z commutes with each element of G_j for every $j = 1, \ldots, n$. Using the metric h_{JK}, we partition the generators t_K into two subsets, the first containing generators of Z and the second containing the generators orthogonal to Z. We multiply the generators of Z by g_0/g and the generators orthogonal to Z by unity. The full set of generators thus transformed, denoted as $\hat{t}_K$, when restricted to the flavor subgroups G_j, satisfy the same commutation relations as t_K.

In formula (5.33) we will need to know the relationship between the traces of $\hat{t}_K$ and t_K. Since the generators orthogonal to Z lie in the Lie algebra of $SL(2n, C)$, their trace is zero, so that we derive

$$\mathrm{Tr}[\hat{t}_K] = \frac{g_0}{g}\,\mathrm{Tr}[t_K] \tag{5.28}$$

The Hermitian $SL(2n, C) \times U(1)$ gauge generators $\hat{t}_K$ are transformed by Λ into

$$t'_K = \Lambda^{+}\hat{t}_K\Lambda \tag{5.29}$$

To simplify formulas, we define $SL(2n, C) \times U(1)$ gauge generators $T_K = (i/2n)t_K$ and $T'_K = (i/2n)t'_K$. Note that both sets of generators $\{T_K\}$ and $\{T'_K\}$ for $K = 0, 1, \ldots, 4n^2 - 1$ form a basis of the Lie algebra of $SL(2n, C) \times U(1)$, which can be represented by $2n \times 2n$ complex matrices. Henceforth it will simplify derivations to represent generators of the gauge group $SL(2n, C) \times U(1)$ as $2n \times 2n$ matrices instead of $4n \times 4n$ matrices [e.g., in the trace formula (5.33)]. The Lie algebra structure constants relative to the two bases $\{T_K\}$ and $\{T'_K\}$ will be denoted as f^L_{JK} and f'^L_{JK}, respectively.

For the gauge group $SL(2n, C) \times U(1)$ the Yang–Mills Lagrangian (2.22) generalizes as follows:

$$L_{g_0} = -\frac{1}{4g}\,\mathrm{Re}[h_{JK}A^J_{\alpha\beta}A^{K\alpha\beta}] + \frac{1}{g_0}\sum_{j=1}^{n}\overline{D_{j\alpha}(\rho_j + \mu)}\,D^{\alpha}_j(\rho_j + \mu) \tag{5.30}$$

where the Yang–Mills coupling constant g is used both in the field tensor $A^L_{\alpha\beta}$, which is defined as

$$A^L_{\alpha\beta} = \partial_\alpha A^L_\beta - \partial_\beta A^L_\alpha + g f'^L_{JK}A^J_\alpha A^K_\beta \tag{5.31}$$

and in the covariant derivatives $D_{j\alpha}$, which are defined as

$$D_{j\alpha} = \partial_\alpha + gA^K_\alpha T'_{jK} \tag{5.32}$$

where $T'_{jK} = \pi_j T'_K \pi_j$ is the projection of the generators T'_K onto the jth flavor Lie subalgebra. Note that this definition of the covariant derivatives $D_{j\alpha}$ ensures that the Euler–Lagrange equation for the Lagrangian (5.30) with the orthogonal constraint (5.22) commutes with each flavor restriction.

The gauge potentials A^K_α and the complex scalar fields ρ_j are defined from the map (5.26) and formula (5.21). As previously stated, the fermion masses in the Lagrangian (5.27) are given by $m_j = \lambda_j m$. In the Lagrangian (5.30) we define the mass parameter μ by $\mu = (2/g_0)m$.

The gauge generators T'_K act on the complex scalars ρ_j and μ by the well-known one-dimensional representation of the Lie algebra of $SL(2n, C) \times U(1)$ given by

$$T'_K \rho_j = \mathrm{Tr}[T'_K] \rho_j \tag{5.33}$$

and similarly for the complex scalar μ. (Note that, as previously stated, the trace is computed in the representation where T'_K is a $2n \times 2n$ matrix.)

By a straightforward derivation from formulas (5.25), (5.28), (5.29), (5.32), and (5.33), the covariant derivatives $D_{j\alpha}$ in the Lagrangian (5.30) act as follows:

$$D_{j\alpha}(\rho_j + \mu) = \partial_\alpha \rho_j + i g_0 \sum_{j=1}^{n} \lambda_j A_\alpha^{(j)} (\rho_j + \mu) \tag{5.34}$$

where we set

$$A_\alpha^{(j)} = -i A_\alpha^K \, \mathrm{Tr}[\pi_j T_K] \tag{5.35}$$

Note that when the map (5.26) is restricted to the jth flavor subspace (i.e., $\Psi = \pi_j \Psi$), then in formula (5.31) by first transforming to the appropriate basis of the Lie algebra (for which the first four generators span the jth flavor Lie subalgebra and the remaining generators are associated with gauge potentials which vanish), and then transforming back to the generators T'_K, we obtain the following expression for the Yang–Mills field tensor $A^L_{\alpha\beta}$:

$$A^L_{\alpha\beta} = \partial_\alpha A^L_\beta - \partial_\beta A^L_\alpha + g \lambda_j f^L_{JK} A^J_\alpha A^K_\beta \tag{5.36}$$

where f^L_{JK} are the Lie algebra structure constants relative to the gauge generators T_K. Similarly, in formula (5.35) we have $A_\alpha^{(j)} = A_\alpha^0$ and $A_\alpha^{(i)} = 0$ for $i \neq j$. Moreover, from formulas (5.15) and (5.21) we have $\rho_j = 0$ for $i \neq j$. Then with both $A_\alpha^{(i)}$ and ρ_i vanishing for $i \neq j$, formula (5.34) gives $D_{i\alpha}(\rho_i + \mu) = 0$ for $i \neq j$. Thus, setting $\rho_j = \rho$ and $D_{j\alpha} = D_\alpha$, we find that formula (5.34) reduces to

$$D_\alpha(\rho + \mu) = \partial_\alpha \rho + i g_0 \lambda_j A_\alpha^0 (\rho + \mu) \tag{5.37}$$

Clearly in formulas (5.36) and (5.37), the Yang–Mills coupling constants g and g_0 have become $g\lambda_j$ and $g_0\lambda_j$ for the jth flavor subspace. Thus, the Lagrangian (5.30) equals the Lagrangian (2.22) with $g\lambda_j$ and $g_0\lambda_j$ instead of g and g_0 as the Yang–Mills coupling constants. Consequently, the jth fermion mass is $m_j = \frac{1}{2} g_0 \lambda_j \mu$ instead of $\frac{1}{2} g_0 \mu$.

The following proposition generalizes formula (2.25) and its implications for the existence of wave packets to multiplets of n bispinor fields.

Proposition 6. The Euler–Lagrange equation for the Yang–Mills Lagrangian (5.30), with the orthogonal constraint (5.22), commutes with each flavor restriction. If we restrict the fields to a single flavor, then

$$L = \lim_{g_0 \to \infty} L_{g_0} \tag{5.38}$$

where L and L_{g_0} are the Lagrangians (5.27) and (5.30), respectively. Furthermore, this limit only exists for wave packets associated with a single fermion flavor, hence only these can propagate as g_0 becomes large.

Note that the phrase "commutes with each flavor restriction" used in the first statement of Proposition 6 means that restricting to single flavor solutions of the Euler–Lagrange equation is equivalent to restricting the fields in the Lagrangian to a single flavor. This implies that all respective observables associated with the Lagrangians (5.38) are equal when restricted to physically realizable single flavor solutions. Thus, as discussed in Section 1, the Lagrangians (5.38) are equivalent for quantum field theory [8].

Proof. When Ψ is restricted to a flavor subspace [associated with the parity map γ^0 and flavor map γ^5; see the definition of flavor subspace following formula (5.9)], all bispinor components of $\Psi = (\Psi_1, \ldots, \Psi_n)$ vanish except for the jth flavor component Ψ_j. It follows from the map (5.26) that the $SL(2n, C) \times U(1)$ gauge potentials A^K_α then vanish except for the $SL(2, R) \times U(1)$ subset associated with the single bispinor field Ψ_j. Inspection of the Yang–Mills Lagrangian L_{g_0}, the orthogonal constraint (5.22), and the resulting Euler–Lagrange equation shows that these equations commute with the action of setting the components of (A^K_α, ρ_j) to zero that are not associated with Ψ_j. Thus formula (5.38) follows from formulas (2.24), (2.25), (5.27), (5.36), and (5.37), and the fact that $\xi = \pi_j \xi$ in formulas (5.15) and (5.20) implies that $s_j = s$ and $s_i = 0$ for $i \neq j$. By the argument preceding formula (2.22), the limit (5.38) only exists for wave packets associated with a single flavor, hence only these can propagate. QED

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